How Pauli-Villars' regularization tells the Nambu-Goto and Polyakov strings apart

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- Y. M. `arXiv:2204.10205 [hep-th]`
- J. Ambjørn, Y. M. MPLA 36, 2150136 (2021) [`arXiv:2103.10259`]
- Y. M. Nucl. Phys. B 967, 115398 (2021) [`arXiv:2102.04753 [hep-th]`]
- Y. M. JHEP 07, 104 (2018) [`arXiv:1802.07541`]
- J. Ambjørn, Y. M. IJMPA 32, 1750187 (2017) [`arXiv:1709.00995`]
- J. Ambjørn, Y. M. Phys. Rev. D 96, 086024 (2017) [`arXiv:1704.03059`]
- J. Ambjørn, Y. M. Phys. Lett. B 770, 352 (2017) [`arXiv:1703.05382`]
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- J. Ambjørn, Y. M. Phys. Rev. D 93, 066007 (2016) [`arXiv:1510.03390`]

Content of the talk

- effective string philosophy
  – invariant cutoff $\boxed{\Lambda^2\sqrt{g}}$

- mean-field solution of regularized bosonic string
  – the Alvarez-Arvis spectrum
  – the Lilliputian scaling limit

- instability of the classical vacuum for $d > 2$
  – stability of the mean-field vacuum for $2 < d < 26$

- systematic expansion about the mean field
  – Pauli-Villars' regularization
  – Theorem about cancellation of infrared divergences

- the beyond Liouville action
  – integrating the Lagrange multiplier λ^{ab}
  – the energy-momentum tensor
  – DDK computation of conformal weight and the central charge
  – R^2 versus minimal Polyakov's string
  – difference between the Nambu-Goto and Polyakov strings
  – cancellation of logs at one and two loops

1. Introduction

Problems of string theory

inherited from 1980's

- Non-perturbative lattice regularization (by dynamical triangulation) scales to a continuum string for $d \leq 1$ but does not for $d > 1$ (same for hypercubic latticization of Nambu-Goto string in $d > 2$)

  Durhuus, Fröhlich, Jonsson (1984), Ambjørn, Durhuus (1987)

- Knizhnik-Polyakov-Zamolodchikov (1988), David (1988), Distler-Kawai (1989)

  string susceptibility index of (closed) Polyakov string is not real for $1 < d < 25$

$$\gamma_{\text{str}} = (1-h)\frac{d-25-\sqrt{(d-1)(d-25)}}{12} + 2 \qquad \boxed{\text{genus } h}$$

Philosophy of effective string

String is formed by more fundamental constituents

Effective or induced or emergent string makes sense when it is long

Examples:

- Abrikosov vortices in superconductor
- Nielsen-Olesen string in the Higgs model
- Confining string in QCD

In particular no tachyon for $\beta > \beta_{\text{tachyon}}$

Pretty much like the view on Quantum Electrodynamics

2. Mean-field vacuum of bosonic string

Nambu-Goto string via Lagrange multiplier

Lagrange multiplier λ^{ab} for independent metric tensor ρ_{ab}

$$K_0 \int \mathrm{d}^2\omega \sqrt{\det \partial_a X \cdot \partial_b X} = K_0 \int \mathrm{d}^2\omega \sqrt{\rho} + \frac{K_0}{2} \int \mathrm{d}^2\omega \, \lambda^{ab} \left(\partial_a X \cdot \partial_b X - \rho_{ab} \right)$$

World-sheet parameters $\omega_1, \omega_2 \in \omega_L \times \omega_\beta$ rectangle

Closed bosonic string winding once around compactified dimension of length β, propagating (Euclidean) time L (cylinder or torus).
No tachyon if β is large enough

Classical solution

$$X^1_{\mathrm{cl}} = \frac{L}{\omega_L}\omega_1, \qquad X^2_{\mathrm{cl}} = \frac{\beta}{\omega_\beta}\omega_2, \qquad X^\perp_{\mathrm{cl}} = 0,$$

$$[\rho_{ab}]_{\mathrm{cl}} = \mathrm{diag}\left(\frac{L^2}{\omega_L^2}, \frac{\beta^2}{\omega_\beta^2} \right) \qquad \lambda^{ab}_{\mathrm{cl}} = \rho^{ab}_{\mathrm{cl}} \sqrt{\rho_{\mathrm{cl}}}$$

minimizes the Nambu-Goto action (a classical vacuum)

Conformal gauge if $\displaystyle \frac{\omega_L}{\omega_\beta} = \frac{L}{\beta}$ then $\lambda^{ab}_{\mathrm{cl}} = \delta^{ab}$

Induced (or emergent) action

Gaussian path integral over X_{q}^{μ} by splitting $X^{\mu}=X_{\mathsf{cl}}^{\mu}+X_{\mathsf{q}}^{\mu}$:

$$S_{\mathsf{ind}}=K_0\int \mathrm{d}^2\omega\,\sqrt{\rho}+\frac{K_0}{2}\int \mathrm{d}^2\omega\,\lambda^{ab}\left(\partial_a X_{\mathsf{cl}}\cdot\partial_b X_{\mathsf{cl}}-\rho_{ab}\right)$$

$$+\frac{d}{2}\operatorname{tr}\log\mathcal{O},\qquad\qquad \mathcal{O}=-\frac{1}{\sqrt{\rho}}\partial_a\lambda^{ab}\partial_b.$$

Operator $\mathcal{O}$ reproduces the Laplacian Δ for $\lambda^{ab}=\rho^{ab}\sqrt{\det\rho}$

Additional ghost determinant in the conformal gauge $\rho_{ab}=\rho\delta_{ab}$

$$-\frac{1}{2}\operatorname{tr}\log\left(-\Delta_a^b+\frac{1}{2}(\Delta_a^b\log\rho)\right)$$

Induced (or emergent) action coincides with effective action for smooth fields

2D determinants diverge and has to be regularized

Regularization of determinants

Proper-time regularization of the trace

$$\operatorname{tr}\log\mathcal{O}\big|_{\text{reg}} = -\int_{a^2}^{\infty}\frac{d\tau}{\tau}\operatorname{tr}e^{-\tau\mathcal{O}}, \qquad \Lambda^2 = \frac{1}{4\pi a^2}$$

Pauli-Villars regularization of the trace Ambjørn, Y.M. (2017)

$$\det(\mathcal{O})\big|_{\text{reg}} \equiv \frac{\det(\mathcal{O})\det(\mathcal{O}+2M^2)}{\det(\mathcal{O}+M^2)^2}$$

$$\operatorname{tr}\log\mathcal{O}\big|_{\text{reg}} = -\int_0^{\infty}\frac{d\tau}{\tau}\operatorname{tr}e^{-\tau\mathcal{O}}\left(1-e^{-\tau M^2}\right)^2, \qquad \Lambda^2 = \frac{M^2}{2\pi}\log 2.$$

is convergent as finite regulator mass M and divergent as $M\to\infty$.

For Pauli-Villars regularization beautiful diagrammatic technique and det's can be exactly computed for certain metrics by the Gel'fand-Yaglom technique to compare with the Seeley expansion

$$\left\langle\omega\right| e^{-\tau\mathcal{O}}\left|\omega\right\rangle = \frac{1}{4\pi\tau}\frac{1}{\sqrt{\det\lambda^{ab}}} + \frac{R}{24\pi} + \mathcal{O}(\tau)$$

which starts with the term $1/\tau$ in 2 dimensions. For $\tau\sim 1/\Lambda^2$ higher terms are suppressed as R/Λ^2 only for smooth fields but revive if not

Mean-field vacuum

Ambjørn, Y.M. (2017)

The result for diagonal and constant $\lambda^{ab} = \bar{\lambda}\delta^{ab}$ and $\rho_{ab} = \bar{\rho}\delta_{ab}$

$$
\begin{aligned}
S_{\text{eff}} &= \frac{K_0}{2}\bar{\lambda}\left(\frac{L^2}{\omega_L^2} + \frac{\beta^2}{\omega_\beta^2}\right)\omega_L\omega_\beta + K_0(1-\bar{\lambda})\bar{\rho}\,\omega_L\omega_\beta \\
&\quad - \left(\frac{d}{2\bar{\lambda}} - 1\right)\Lambda^2\bar{\rho}\,\omega_L\omega_\beta - \frac{\pi(d-2)}{6}\frac{\omega_L}{\omega_\beta}
\end{aligned}
$$

for $L \gg \beta$ omitting the boundary terms.
The minimum is reached at (quantum vacuum)

$$
\begin{aligned}
\bar{\lambda} &= \frac{1}{2}\left(1 + \frac{\Lambda^2}{K_0} + \sqrt{\left(1 + \frac{\Lambda^2}{K_0}\right)^2 - \frac{2d\Lambda^2}{K_0}}\right) \\
\bar{\rho} &\propto \frac{\bar{\lambda}}{\sqrt{(1 + \frac{\Lambda^2}{K_0})^2 - \frac{2d\Lambda^2}{K_0}}} \\
\omega_\beta &= \frac{\omega_L}{L}\sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0\bar{\lambda}}} \\
S_{\text{mf}} &= K_0\bar{\lambda}L\sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0\bar{\lambda}}}
\end{aligned}
$$

Mean-field vacuum (cont.)

The approximation describes a mean field which takes into account an infinite set of pertubative diagrams about the classical vacuum. Then λ^{ab} and ρ_{ab} do not fluctuate which becomes exact at large d.

It is like 2d $O(N)$ sigma-model at large N where the Lagrange multiplier does not fluctuate (summing the bubble graphs). The large-N vacuum is very closed to the physical vacuum even for $N = 3$.

The minimization over ω_β/ω_L is also needed at the saddle point.

The square root is well-defined if

$$K_0 > K_* = \left(d - 1 + \sqrt{d^2 - 2d}\right)\Lambda^2 \stackrel{d\to\infty}{\to} 2d\Lambda^2$$

Perturbation theory is recovered by expanding in $1/K_0 \sim \hbar$. Then $\bar{\lambda}$ ranges between 1 (classical) and (quantum) value

$$\bar{\lambda}_* = \frac{1}{2}\left(d - \sqrt{d^2 - 2d}\right) \stackrel{d\to\infty}{\to} \frac{1}{2}$$

3. Two scaling regimes: Gulliver's vs. Lilliputian

Lattice-like scaling limit (Gulliver's)

The ground state energy (Alvarez-Arvis)

$$E_0(\beta) = K_0\bar\lambda\sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0\bar\lambda}}$$

does not scale because $K_0 > K_* \sim \Lambda^2$ for $\bar\lambda$ to be real ($> \bar\lambda_*$). Let

$$\beta^2 > \beta^2_{\rm min} = \frac{\pi(d-2)}{3K_*\bar\lambda_*}, \qquad \bar\lambda_* = \frac{1}{2}\left(d - \sqrt{d^2-2d}\right)$$

for not to have a tachyon.
Choose the smallest possible value $\beta = \beta_{\rm min}$

$$E_0(\beta) \propto \frac{K_0\bar\lambda}{\Lambda}\sqrt{\bar\lambda - \bar\lambda_*}$$

which scales to m if

$$\bar\lambda - \bar\lambda_* \propto \frac{m^2}{\Lambda^2}, \qquad K_0 - K_* \propto \frac{m^4}{\Lambda^2}$$

The scaling does not exist for excited states (larger values of β) and thus is particle-like similar to lattice regularizations of a string, where only the lowest mass scales to finite, excitations scale to infinity

Durhuus, Fröhlich, Jonsson (1984), Ambjørn, Durhuus (1987)

Lattice-like scaling limit (Gulliver's)

The ground state energy

$$E_N(\beta) = K_0\bar{\lambda}\sqrt{\beta^2 + \frac{1}{K_0\bar{\lambda}}\left(-\frac{\pi(d-2)}{3} + 8N\right)}$$

does not scale because $K_0 > K_* \sim \Lambda^2$ for $\bar{\lambda}$ to be real ($> \bar{\lambda}_*$). Let

$$\beta^2 > \beta^2_{\text{min}} = \frac{\pi(d-2)}{3K_*\bar{\lambda}_*}, \qquad \bar{\lambda}_* = \frac{1}{2}\left(d - \sqrt{d^2 - 2d}\right)$$

for not to have a tachyon.
Choose the smallest possible value $\beta = \beta_{\text{min}}$

$$E_0(\beta) \propto \frac{K_0\bar{\lambda}}{\Lambda}\sqrt{\bar{\lambda} - \bar{\lambda}_*}$$

which scales to m if

$$\bar{\lambda} - \bar{\lambda}_* \propto \frac{m^2}{\Lambda^2}, \qquad K_0 - K_* \propto \frac{m^4}{\Lambda^2}$$

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Durhuus, Fröhlich, Jonsson (1984), Ambjørn, Durhuus (1987)

Lilliputian string-like scaling limit

Let us "renormalize" the units of length

$$L_R = \sqrt{\frac{\bar{\lambda}}{\bar{\lambda} - \bar{\lambda}_*}}\, L, \qquad \beta_R = \sqrt{\frac{\bar{\lambda}}{\bar{\lambda} - \bar{\lambda}_*}}\, \beta$$

to obtain for the effective action

$$S_{\mathrm{mf}} = K_R\, L_R \sqrt{\beta_R^2 - \frac{\pi d}{3K_R}}, \qquad K_R = K_0(\bar{\lambda} - \bar{\lambda}_*)$$

The renormalized string tension K_R scales to finite if

$$K_0 \to K_* + \frac{K_R^2}{2\Lambda^2\sqrt{d^2 - 2d}}, \qquad K_* = \left(d - 1 + \sqrt{d^2 - 2d}\right)\Lambda^2$$

reproducing the Alvarez-Arvis spectrum of continuum string.

The average area is also finite

$$\langle Area \rangle = L_R \frac{\left(\beta_R^2 - \frac{\pi d}{6K_R}\right)}{\sqrt{\beta_R^2 - \frac{\pi d}{3K_R}}}$$

$\Rightarrow$ minimal area for $\beta_R^2 \gg \pi d/(3K_R)$ and diverges if $\beta_R^2 \to \pi d/(3K_R)$

Lilliputian string-like scaling limit (cont.)

It looks like for the zeta-function regularization, but

$$\text{length} = \sqrt{\frac{\bar\lambda - \bar\lambda_*}{\bar\lambda}}\,\text{length}_R \propto \frac{\sqrt{K_R}}{\Lambda}\,\text{length}_R$$

in target space which is of order of the cutoff ($\Rightarrow$ Lilliputian)

Nevertheless, the cutoff (in parameter space) $\boxed{\Delta\omega = 1/(\Lambda\sqrt[4]{g})}$ fixes maximal number of modes in the mode expansion

$$X_q = \sum_{m,n\geq 0}\left(a_{mn}\cos\frac{2\pi m\omega_2}{\omega_\beta} + b_{mn}\sin\frac{2\pi m\omega_2}{\omega_\beta}\right)\sin\frac{\pi n\omega_1}{\omega_L},$$

to be

$$n^{(1)}_{\max} \sim \Lambda\sqrt[4]{g}\,\omega_L \qquad n^{(2)}_{\max} \sim \Lambda\sqrt[4]{g}\,\omega_\beta$$

Classically $\sqrt[4]{g}\,\omega_\beta = \beta$ reproducing Brink-Nielsen (1973)

Quantumly $\sqrt[4]{g}\,\omega_\beta \propto \dfrac{\beta}{\sqrt{\bar\lambda - \bar\lambda_*}} = \dfrac{\sqrt{K_0}\,\beta}{\sqrt{K_R}}$ is much larger

$\Longrightarrow$ classical music can be played on the Lilliputian strings

The Lilliputian world

- The Lilliputian world is of the size of the target-space cutoff
- It is still a continuum because infinitely smaller distances can be resolved (infinitely many stringy modes)
- The Lilliputian scaling regime is perfectly recovered by results of the zeta-function regularization
- Linear Regge trajectories signalize about the Lilliputian world
- Gulliver's tools are too coarse to resolve the Lilliputian world (this is why lattice string regularizations of 1980's never reproduce canonical quantization)

The KING of BROBDINGNAG, and GULLIVER.

—Vide. *Swift's Gulliver*: Voyage to Brobdingnag.

Pub^d. June 26^th 1803. by H. Humphrey 27 St. James's Street.

4. Instability of classical vacuum

Semiclassical energy

Brink, Nielsen (1973)

Semiclassical (or one-loop) correction due to zero-point fluctuations

$$S_{1l} = \left[K_0 - \frac{(d-2)}{2}\Lambda^2\right] L\beta - \frac{\pi(d-2)L}{6\beta}$$

bulk term Casimir energy

To make it finite, it is introduced the renormalized string tension

$$K_R = K_0 - \frac{(d-2)}{2}\Lambda^2$$

which is kept finite as $\Lambda \to \infty$. Then it is assumed that it works order by order of the perturbative expansion about the classical vacuum, so that K_R can be made finite by fine tuning K_0.

We see however from the mean-field formula

$$S_{\text{mf}} = K_0\bar{\lambda}L\sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0\bar{\lambda}}}$$

that S_{mf} never vanishes with changing K_0 (except ...).
Thus the one-loop correction simply lowers for $d > 2$ the energy of the classical vacuum state which may indicate its instability.

Effective potential

To investigate stability of the vacuum, add the source term like in QFT

$$S_{\mathrm{src}} = \frac{K_0}{2} \int \mathrm{d}^2\omega\, j^{ab} \rho_{ab}$$

defining the field

$$\rho_{ab}(j) = -\frac{2}{K_0} \frac{\delta}{\delta j^{ab}} \log Z.$$

Minimizing for constant $j^{ab} = j\delta^{ab}$ we find Ambjørn, Y.M. (2017)

$$\bar{\lambda}(j) = \frac{1}{2}\left(1 + j + \frac{\Lambda^2}{K_0}\right) + \sqrt{\frac{1}{4}\left(1 + j + \frac{\Lambda^2}{K_0}\right)^2 - \frac{d\Lambda^2}{2K_0}}$$

$$\bar{\rho}(j) = \frac{1}{2} + \frac{1 + j + \frac{\Lambda^2}{K_0}}{\sqrt{\left(1 + j + \frac{\Lambda^2}{K_0}\right)^2 - \frac{2d\Lambda^2}{K_0}}} \qquad \bar{\lambda}(\bar{\rho}) = \sqrt{\frac{d\Lambda^2}{2K_0}} \sqrt{\frac{\bar{\rho}}{\bar{\rho} - 1}}$$

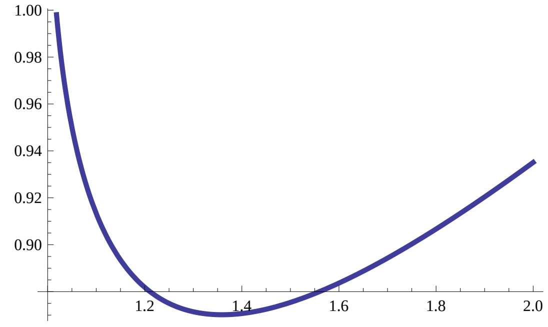
in the mean field approximation for $\omega_L = L$ and $\omega_\beta = \beta \gg 1\sqrt{K_0}$,

Effective potential (cont.)

"Effective potential" is given by the Legendre transformation

$$\Gamma(\bar{\rho}) = -\frac{1}{K_0 L \beta} \log Z - j(\bar{\rho})\bar{\rho}$$

In the mean-field approximation



$$\Gamma(\bar{\rho}) = \left(1 + \frac{\Lambda^2}{K_0}\right)\bar{\rho} - \sqrt{\frac{2d\Lambda^2}{K_0}\bar{\rho}(\bar{\rho}-1)}$$

Classical vacuum $\bar{\rho} = 1$ is unstable and a stable minimum occurs at

$$\bar{\rho}(0) = \frac{1}{2} + \frac{1 + \frac{\Lambda^2}{K_0}}{2\sqrt{\left(1 + \frac{\Lambda^2}{K_0}\right)^2 - \frac{2d\Lambda^2}{K_0}}}$$

if $K_0 > K_*$ (same value as before for $\beta \gg 1/\sqrt{K_0}$).
Near the minimum

$$\Gamma(\bar{\rho}) = \left[\left(1 + \frac{\Lambda^2}{K_0}\right)^2 - \frac{2d\Lambda^2}{K_0}\right]^{1/2} + \frac{K_0}{2d\Lambda^2}\left[\left(1 + \frac{\Lambda^2}{K_0}\right)^2 - \frac{2d\Lambda^2}{K_0}\right]^{3/2} [\bar{\rho} - \bar{\rho}(0)]^2$$

String susceptibility

Entropy of surfaces of the area A

$$\left\langle \delta^{(1)}\left(\int \rho - A\right)\right\rangle \overset{A\to\infty}{\propto} A^{\gamma_{\rm str}-2}\,\mathrm{e}^{CA}$$

Passage from grand canonical to canonical ensemble at fixed area A.

Representing

$$\delta^{(1)}\left(\int \rho - A\right) = \int_{\uparrow} \frac{\mathrm{d}j}{2\pi\mathrm{i}}\,\mathrm{e}^{jA - j\int\rho}$$

we identify j with the source above.

Expanding about the saddle point Ambjørn, Y.M. (2017), (2021)

$$j\rho - \bar{\lambda}(j) = -\Gamma(\rho) + \sqrt{\frac{2K_0}{d\Lambda^2}}\,[\rho(\rho-1)]^{3/2}\,(\Delta j)^2$$

$\Longrightarrow$ Helmholtz free energy in the mean-field approximation

$$F = K_0 L\beta\Gamma(\frac{A}{L\beta}) + \frac{3}{4}\log\left[\frac{A}{L\beta}\left(\frac{A}{L\beta}-1\right)\right] + \text{const.}$$

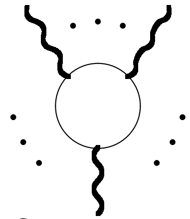
$\Longrightarrow$ thus $\boxed{\gamma_{\rm str} = 1/2}$. It is quite different from $\boxed{\gamma_{\rm str} = 1}$ of KPZ-DDK

5. Stability of mean-field vacuum

Coleman-Weinberg potential

Integrating out X_q^μ we get (a part of) the effective action

$$\frac{d}{2}\,\mathrm{tr}\ln\left[-\frac{1}{\rho}\partial_a\lambda^{ab}\partial_b\right]_{\mathrm{reg}} \quad = \quad \sum_n \frac{1}{n}$$



wavy lines correspond to fluctuations $\delta\lambda^{ab}$ or $\delta\rho$ about ground state

$$\lambda^{ab}(\omega) = \bar{\lambda}\delta^{ab} + \delta\lambda^{ab}, \qquad \rho(\omega) = \bar{\rho} + \delta\rho$$

Positively definite quadratic form for imaginary $\delta\lambda^{ab}$ and real $\delta\rho$

$$S^{(2)}_{\mathrm{div}} = \int\left[-\frac{d\Lambda^2\bar{\rho}}{2\bar{\lambda}}\delta\lambda_2 - \left(K_0 - \frac{d\Lambda^2}{2\bar{\lambda}^2}\right)\delta\rho\frac{\delta\lambda^{aa}}{2} - \frac{d\Lambda^2\bar{\rho}}{2\bar{\lambda}^3}\left(\frac{\delta\lambda^{aa}}{2}\right)^2\right]$$

$$\delta\lambda_2 = \frac{1}{8\bar{\lambda}}(\delta\lambda_{11} - \delta\lambda_{22})^2 + \frac{1}{2\bar{\lambda}}(\delta\lambda_{12})^2$$

Polyakov's book: typical $\delta\lambda \sim 1/\Lambda$ so λ^{ab} is localized and decouples.
Thus only ρ fluctuates (stable for $2 < d < 26$)

The private life occurs at distances $\sim \Lambda^{-1}$ but is observable Y.M. (2021)

6. Fluctuations about mean field

Expansion about the mean field

For $2 < d < 26$ define the partition function

$$Z[b_0] = \int \mathcal{D}\rho \, \mathrm{e}^{-S_{\text{ind}}}$$

$b_0^2 = 6/(26-d)$ controls a "semiclassical" expansion about the mean field which plays the role of a "classical" vacuum as $b_0^2 \to 0$.
Like $1/N$ expansion in the sigma model)
(ghosts differ the situation from the sigma model)

Massive determinants of Pauli-Villars' regulators (ghost statistics)

$$\det\left(-\frac{\bar{\lambda}}{\rho}\partial^2 + M^2\right)^{d/2} = \int \mathcal{D}Y_M^\mu \, \mathrm{e}^{-\frac{K_0}{2}\int \mathrm{d}^2\omega \left(\bar{\lambda}\delta^{ab}\partial_a Y_M \cdot \partial_b Y_M + M^2 \rho Y_M \cdot Y_M\right)}$$

contribute to the energy-momentum tensor

$$T(z) \equiv T_{zz} = 2\pi K_0 \bar{\lambda} \partial_z X \cdot \partial_z X + \text{ghosts} + \text{regulators}$$

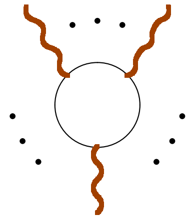
Regulators explicitly interact with the metric ρ in conformal gauge

$$\left\langle \delta\rho(-p) Y_M^\mu(k+p) Y_M^\nu(-k) \right\rangle_{\text{truncated}} = -K_0 M^2 \delta^{\mu\nu}$$

Computation of S_{eff}

Regularized determinant

$$\frac{d}{2}\,\mathrm{tr}\ln\left[-\frac{\bar{\lambda}}{\rho}\partial^2\right]_{\text{reg}} = \sum_n \frac{1}{n}$$



wavy lines correspond to

$$\delta\rho(z) = \bar{\rho}\left(\mathrm{e}^{\varphi(z)} - 1\right)$$

Terms of all order in φ but tremendous cancellation for smooth φ:

$$S_{\text{eff}} = \frac{26-d}{96\pi}\int \partial_a\varphi\partial_a\varphi \qquad \text{if} \quad R \ll \Lambda^2$$

Exact result when quantum fluctuations of φ disregarded

In general we expand $\varphi = \varphi_{\text{cl}} + \varphi_{\text{q}}$ and average over φ_{q}

Fiducial metric ($\hat{R} \ll \Lambda^2$)

$$\hat{\rho}_{ab} = \delta_{ab} e^{\varphi_{\text{cl}}}$$

Path integrating over λ^{ab}

Simplified quadratic action $(\lambda^{z\bar{z}} = 0)$

$$\mathcal{S}^{(2)} = \int \left[\frac{1}{4\pi b_0^2}\partial\varphi\bar{\partial}\varphi + \nu\left(\lambda^{zz}\nabla\partial\varphi + \lambda^{\bar{z}\bar{z}}\bar{\nabla}\bar{\partial}\varphi\right) - d\Lambda^2\bar{\rho}\,\mathrm{e}^{\varphi}\lambda^{zz}\lambda^{\bar{z}\bar{z}}\right]$$

Integrating out λ^{zz} and $\lambda^{\bar{z}\bar{z}}$

$$\begin{aligned}\mathcal{S}^{(2)} &= \int \left[\frac{1}{4\pi b_0^2}\partial\varphi\bar{\partial}\varphi + \frac{\nu^2}{d\Lambda^2\bar{\rho}}\mathrm{e}^{-\varphi}(\nabla\partial\varphi)(\bar{\nabla}\bar{\partial}\varphi)\right] \\ &= \int \left[\frac{1}{4\pi b_0^2}\partial\varphi\bar{\partial}\varphi + \frac{\nu^2}{d\Lambda^2\bar{\rho}}\mathrm{e}^{-\varphi}\left(\partial^2\varphi - (\partial\varphi)^2\right)\left(\bar{\partial}^2\varphi - (\bar{\partial}\varphi)^2\right)\right]\end{aligned}$$

Integrating by parts

$$\mathcal{S}^{(2)} = \frac{1}{4\pi b_0^2}\int \left\{\partial\varphi\bar{\partial}\varphi + 4\varepsilon\,\mathrm{e}^{-\varphi}\left[(\partial\bar{\partial}\varphi)^2 + \partial\varphi\bar{\partial}\varphi\partial\bar{\partial}\varphi\right]\right\}, \qquad \varepsilon = \frac{\pi\nu^2 b_0^2}{d\Lambda^2\bar{\rho}}$$

The second additional term does not appear for Polyakov's string

Beyond Liouville action

Integrating over X^μ, ghosts, regulators Y^μ, $\bar{Y}^\mu$, Z^μ and λ^{ab}

$$\mathcal{S} = \frac{1}{16\pi b_0^2} \int \sqrt{g} \left[-R \frac{1}{\Delta} R + 2m_0^2 + a^2 R \left(R + G g^{ab} \, \partial_a \frac{1}{\Delta} R \, \partial_b \frac{1}{\Delta} R \right) \right]$$

curvature squared R^2 (Polyakov) + nonlocal $G \neq 0$ (Nambu-Goto)

or

$$\mathcal{S} = \frac{1}{4\pi b_0^2} \int [\partial\varphi\bar{\partial}\varphi + \frac{\mu_0^2}{2} \mathrm{e}^{\varphi} + 4\varepsilon \, \mathrm{e}^{-\varphi} (\partial\bar{\partial}\varphi)^2 - 4G\varepsilon \, \mathrm{e}^{-\varphi} \partial\varphi\bar{\partial}\varphi\partial\bar{\partial}\varphi]$$

in conformal gauge $\rho_{ab} = \delta_{ab}\bar{\rho}\,\mathrm{e}^{\varphi}$ with worldsheet cutoff $\varepsilon = a^2/\bar{\rho}$ and $\mu_0^2 = m_0^2 \bar{\rho}$

Classically extra terms vanishes for smooth $\varepsilon R \ll 1$ but
quartic derivative provides UV cutoff but also interaction with
coupling $\varepsilon \Rightarrow$ uncertainties $\varepsilon \times \varepsilon^{-1}$ so they revive quantumly
$\Rightarrow$ produce anomalies

Beyond Liouville action (cont.)

Illustration by the Seeley expansion of $\mathrm{e}^{a^2\Delta}$ DeWitt (1963)

$$\left\langle \omega \right| \mathrm{e}^{a^2\Delta} \left| \omega \right\rangle = \frac{1}{4\pi a^2} + \frac{1}{24\pi} R(\omega) + \frac{a^2}{120\pi}\left(\Delta R(\omega) + \frac{1}{2}R^2(\omega)\right) + \ldots$$

Splitting $\varphi = \varphi_{\mathrm{cl}} + \varphi_{\mathrm{q}}$ and averaging over φ_{q}

$$\begin{aligned}\left\langle \phi_{\mathrm{q}}(\omega)\partial^2\phi_{\mathrm{q}}(\omega)\right\rangle_{\mathrm{q}} &= \frac{2b_0^2}{\varepsilon} + \text{less singular} \\ &= \frac{2b_0^2}{a^2}\mathrm{e}^{\phi_{\mathrm{cl}}} + \text{less singular}\end{aligned}$$

we get

$$\langle R(\omega)\rangle_{\mathrm{q}} = \frac{2b_0^2}{a^2} + \text{less singular},$$

$$\left\langle R^2(\omega)\right\rangle_{\mathrm{q}} = \frac{3b_0^2}{\pi}\frac{1}{a^4} + \frac{4b_0^2}{a^2}R_{\mathrm{cl}}(\omega) + \text{less singular}$$

giving contribution to conformal anomaly

7. CFT á la KPZ-DDK

Review of KPZ-DDK

Knizhnik-Polyakov-Zamolodchikov (1988), David (1988), Distler-Kawai (1989)

Liouville action in fiducial (or background) metric $\widehat{g}_{ab}$

$$S_L = \frac{1}{8\pi b^2}\int \sqrt{\widehat{g}}\left(\frac{1}{2}\widehat{g}^{ab}\partial_a\varphi\partial_b\varphi + q\widehat{R}\varphi\right) + \mu^2\int\sqrt{\widehat{g}}\,\mathrm{e}^{\varphi}$$

$$b^2 = b_0^2 + \mathcal{O}(b_0^4), \qquad q = 1 + \mathcal{O}(b_0^2) \qquad b_0^2 = \frac{6}{26-d}$$

are "renormalized" parameters of the effective action.
Energy-momentum pseudotensor

$$T(z) = \text{matter} + \text{ghosts} - \frac{1}{4b^2}\left(\partial_z\varphi\partial_z\varphi - 2q\partial_z^2\varphi\right)$$

Background independence: the total central charge

$$d - 26 + 1 + 6\frac{q^2}{b^2} = 0$$

and the conformal weight

$$\Delta(\,\mathrm{e}^{\varphi}) = q - b^2 = 1$$

$$\Longrightarrow \qquad b = \sqrt{\frac{25-d}{24}} - \sqrt{\frac{1-d}{24}}, \qquad q = 1 + b^2$$

Energy-momentum tensor

For minimal coupling to gravitational $\hat{g}_{ab}$

$$
\begin{aligned}
-4b_0^2 T_{ab}^{(\mathrm{min})} = {} & \partial_a\varphi\partial_b\varphi - \frac{1}{2}g_{ab}\partial^c\varphi\partial_c\varphi - \mu_0^2 g_{ab} - \varepsilon\partial_a\varphi\partial_b\Delta\varphi - \varepsilon\partial_a\Delta\varphi\partial_b\varphi \\
& + \varepsilon g_{ab}\partial^c\varphi\partial_c\Delta\varphi + \frac{\varepsilon}{2}g_{ab}(\Delta\varphi)^2 - G\varepsilon\partial_a\varphi\partial_b\varphi\Delta\varphi + G\frac{\varepsilon}{2}\partial_a\varphi\partial_b(\partial^c\varphi\partial_c\varphi) \\
& + G\frac{\varepsilon}{2}\partial_a(\partial^c\varphi\partial_c\varphi)\partial_b\varphi - G\frac{\varepsilon}{2}g_{ab}\partial^c\varphi\partial_c(\partial^d\varphi\partial_d\varphi)
\end{aligned}
$$

For diffeomorphism invariant action

$$
\begin{aligned}
-4b_0^2 T_{ab} &= -4b_0^2 T_{ab}^{(\mathrm{min})} - 2(\partial_a\partial_b - g_{ab}\partial^c\partial_c)(\varphi - \varepsilon\Delta\varphi + G\frac{\varepsilon}{2}g^{ab}\partial_a\varphi\partial_b\varphi) \\
& \quad + 2G\varepsilon(\partial_a\partial_b - g_{ab}\partial^c\partial_c)\frac{1}{\Delta}\partial^d(\partial_d\varphi\Delta\varphi)
\end{aligned}
$$

It is conserved and traceless (!) thanks to diffeomorphism invariance

$$
\begin{aligned}
-4b_0^2 T_{zz} &= (\partial\varphi)^2 - 2\varepsilon\partial\varphi\partial\Delta\varphi - 2\partial^2(\varphi - \varepsilon\Delta\varphi) - G\varepsilon(\partial\varphi)^2\Delta\varphi \\
& \quad + 4G\varepsilon\partial\varphi\partial(\mathrm{e}^{-\varphi}\partial\varphi\bar{\partial}\varphi) - 4G\varepsilon\partial^2(\mathrm{e}^{-\varphi}\partial\varphi\bar{\partial}\varphi) + G\varepsilon\partial(\partial\varphi\Delta\varphi) \\
& \quad + G\varepsilon\frac{1}{\bar{\partial}}\partial^2(\bar{\partial}\varphi\Delta\varphi)
\end{aligned}
$$

Pauli-Villars' regulators

Pauli-Villars' regulators: Grassmann Y, $\bar{Y}$ (M^2) and normal Z ($2M^2$)

$$\mathcal{S}_{\text{reg.}} = \frac{1}{16\pi b_0^2}\int\sqrt{g}\left[g^{ab}\partial_a Y\partial_b Y + M^2Y^2 + \varepsilon(\Delta Y)^2 + G\varepsilon g^{ab}\partial_a Y\partial_b Y R\right]$$

or in conformal gauge

$$\mathcal{S}_{\text{reg.}} = \frac{1}{4\pi b_0^2}\int\left[\partial Y\bar{\partial}Y + \frac{M^2}{4}\,\mathrm{e}^{\varphi}Y^2 + 4\varepsilon\,\mathrm{e}^{-\varphi}(\partial\bar{\partial}Y)^2 - 4G\varepsilon\,\mathrm{e}^{-\varphi}\partial Y\bar{\partial}Y\partial\bar{\partial}\varphi\right]$$

Conserved and traceless (!) energy-momentum tensor

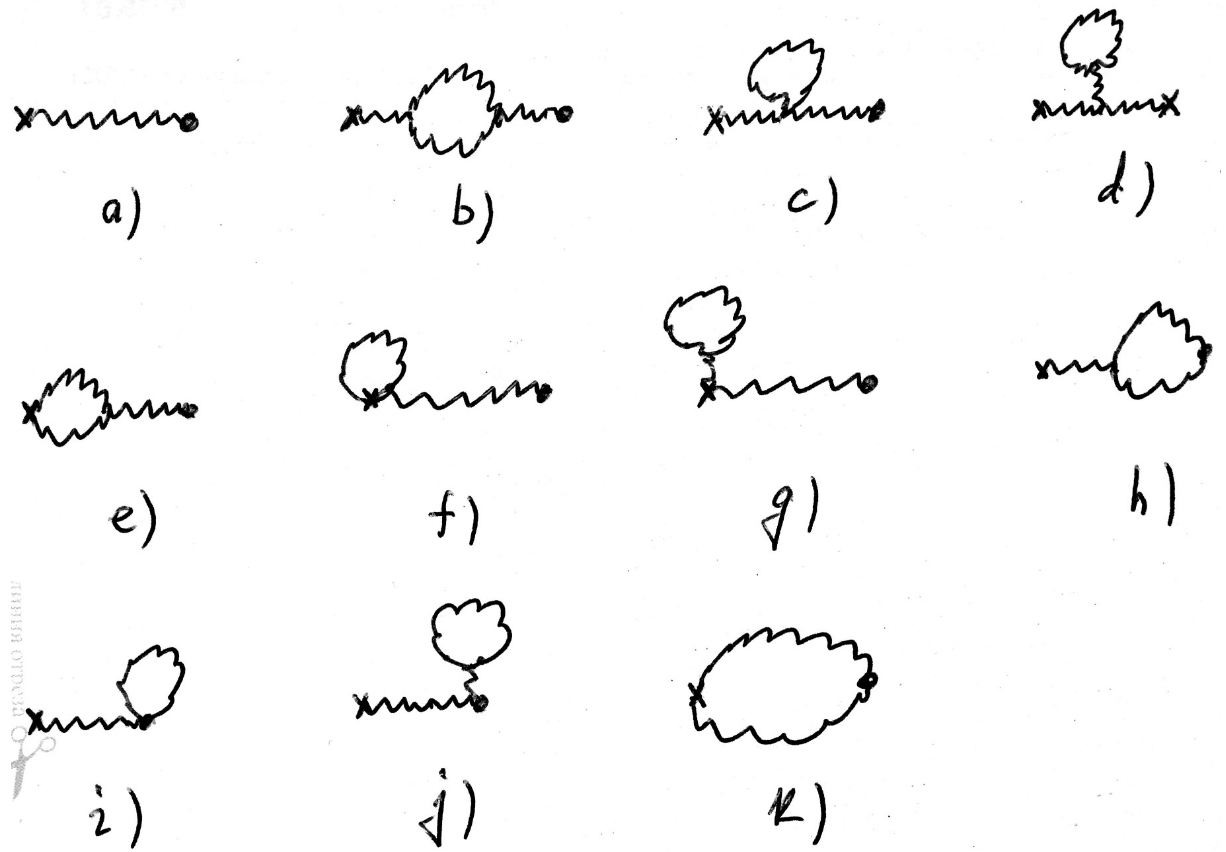
$$\begin{aligned}-4b_0^2T_{ab}^{(\text{reg})} = {} & \partial_a Y\partial_b Y - \frac{1}{2}g_{ab}\partial^c Y\partial_c Y - \frac{M^2}{2}g_{ab}Y^2 - \varepsilon\partial_a Y\partial_b\Delta Y \\ & -\varepsilon\partial_a\Delta Y\partial_b Y + \varepsilon g_{ab}\partial^c Y\partial_c\Delta Y + \frac{\varepsilon}{2}g_{ab}(\Delta Y)^2 - G\varepsilon\partial_a Y\partial_b Y\Delta\varphi \\ & +G\frac{\varepsilon}{2}\partial_a\varphi\partial_b(\partial^c Y\partial_c Y) + G\frac{\varepsilon}{2}\partial_a(\partial^c Y\partial_c Y)\partial_b\varphi - G\frac{\varepsilon}{2}g_{ab}\partial^c\varphi\partial_c(\partial^d Y\partial_d Y) \\ & -G\varepsilon(\partial_a\partial_b - g_{ab}\partial^c\partial_c)(\partial^c Y\partial_c Y).\end{aligned}$$

$\Rightarrow$ conformal invariance expected to be maintained quantumly

$$\begin{aligned}-4b_0^2T_{zz}^{(\text{reg})} &= \partial Y\partial Y - 2\varepsilon\partial Y\partial\Delta Y - G\varepsilon\partial Y\partial Y\Delta\varphi + 4G\varepsilon\partial\varphi\partial(\,\mathrm{e}^{-\varphi}\partial Y\bar{\partial}Y) \\ &\quad -4G\varepsilon\partial^2(\,\mathrm{e}^{-\varphi}\partial Y\bar{\partial}Y)\end{aligned}$$

DDK revisited

One-loop operator products $T_{zz}(z)\, e^{\varphi(0)}$ and $T_{zz}(z)T_{zz}(0)$



Diagrams a) to j) contribute $q\alpha$ to the conformal weight of $e^{\alpha\varphi(0)}$
Diagrams k) contributes $-b^2\alpha^2$ to the conformal weight of $e^{\alpha\varphi(0)}$
Additional terms do not contribute as $\varepsilon \to 0$, reproducing DDK

$$q\alpha - b^2\alpha^2 = 1$$

One-loop central charge

Diagrams a) to j) contribute $6q^2/b^2$ to the central charge as usual.
Diagram k) contributes usual 1 to the central charge but
the nonlocal term in T_{zz} revives

$$2 \cdot \frac{1}{16} \left\langle 2G\varepsilon \partial^3 \frac{1}{\bar\partial} \bar\partial\varphi(z)\bar\partial\varphi(z) \partial\varphi(0)\partial\varphi(0) \right\rangle = (8\pi)^2 \frac{G\varepsilon}{2} \partial^3 \frac{1}{\bar\partial} (\partial\bar\partial G_\varepsilon(z))^2$$
$$\to \frac{G\pi}{2} \partial^3 \frac{1}{\bar\partial} \delta^{(2)}(z) = -3G \frac{1}{z^4},$$

$$2 \cdot \frac{1}{16} \left\langle 2G\varepsilon \partial^3 \frac{1}{\bar\partial} \bar\partial\varphi(z)\bar\partial\varphi(z) (-8\varepsilon) \partial\varphi(0)\partial^2\bar\partial\varphi(0) \right\rangle$$
$$= -(8\pi)^2 4G\varepsilon^2 \partial^3 \frac{1}{\bar\partial} (\partial\bar\partial G_\varepsilon(z) \partial^2\bar\partial^2 G_\varepsilon(z)) \to -G\pi\partial^3 \frac{1}{\bar\partial} \delta^{(2)}(z) = 6G \frac{1}{z^4}$$

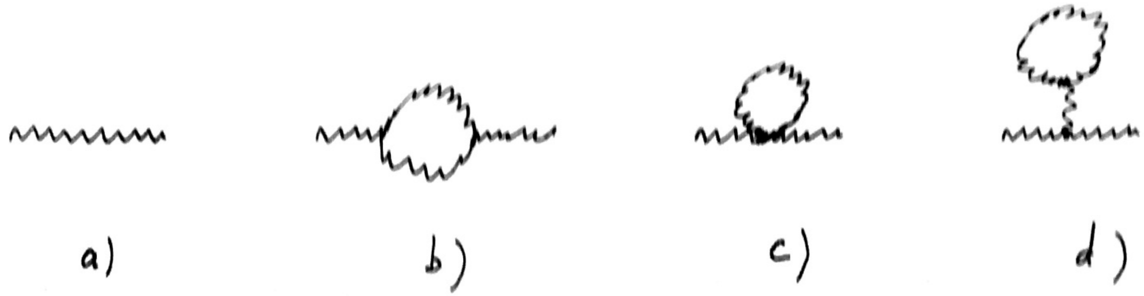
The second DDK equation is modified (assuming one loop is exact)

$$-\frac{6}{b_0^2} + 1 + \frac{6q^2}{b^2} + 6Gq = 0 \quad \Rightarrow \quad \alpha b = \sqrt{\frac{25 - d - 6Gq}{24}} - \sqrt{\frac{1 - d + 6Gq}{24}}$$

8. Algebraic check of DDK

One-loop propagator

One-loop propagator $\langle\varphi(-p)\varphi(p)\rangle$ for $\varepsilon = 0$



$$\text{b)} \;=\; \frac{1}{4}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\left\{\frac{2M^4}{(k^2+M^2)[(k-p)^2+M^2]} - \frac{4M^4}{(k^2+2M^2)[(k-p)^2+2M^2]}\right\}|\varphi(p)|^2 \to -\frac{p^2}{96\pi}|\varphi(p)|^2$$

$\Rightarrow$ the usual conformal anomaly

$$\text{c)} = -\frac{1}{4}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\left\{\frac{2M^2}{(k^2+M^2)} - \frac{2M^2}{(k^2+2M^2)}\right\}|\varphi(p)|^2 = -\frac{M^2}{8\pi}\log 2\,|\varphi(p)|^2$$

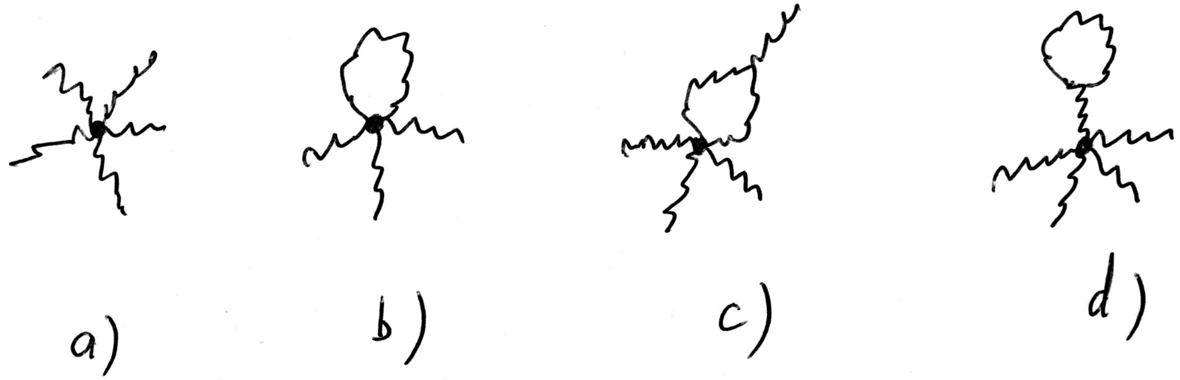
$\Rightarrow$ the renormalization of μ^2 rather than b^2.

One-loop renormalization of b^2

$$\frac{1}{b^2} = \frac{1}{b_0^2} - \frac{1}{6} + \mathcal{O}(b_0^2)$$

One-loop renormalization of α

One-loop renormalization of $\mathrm{e}^{\varphi(0)} \to \mathrm{e}^{\alpha\varphi(0)}$ for $\varepsilon = 0$



$$\mathrm{b)} = \frac{1}{2}\left\langle \varphi(0)^2 \right\rangle = b_0^2\,[\varphi(0) + \text{const.}]$$

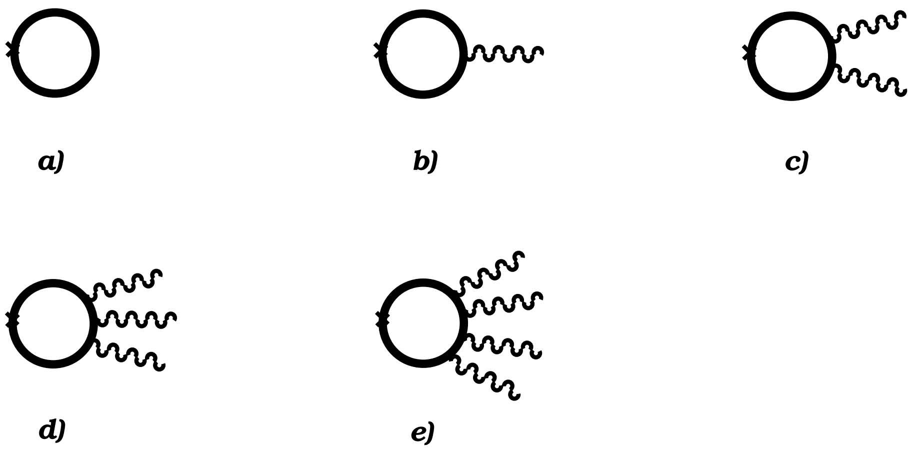
because the propagator logarithmically diverges at coinciding points and the worldsheet cutoff $\varepsilon \propto a^2\,\mathrm{e}^{-\varphi}$.

Standard one-loop renormalization

$$\alpha = 1 + b_0^2 + \mathcal{O}(b_0^4).$$

One-loop renormalization of T_{zz}

One-loop renormalization of T_{zz} for $\varepsilon = 0$ wavy $= \mathrm{e}^{\varphi} - 1$



$$
\text{b)} \;=\; \frac{1}{4}(8\pi)\int \frac{\mathrm{d}^2 k}{(2\pi)^2} k_z (p-k)_z \Big\{ \frac{2M^2}{(k^2+M^2)[(k-p)^2+M^2]} - \frac{2M^2}{(k^2+2M^2)[(k-p)^2+2M^2]} \Big\} \varphi(p) \to \frac{p_z^2}{12}\varphi(p)
$$

Sum of this and the other diagrams

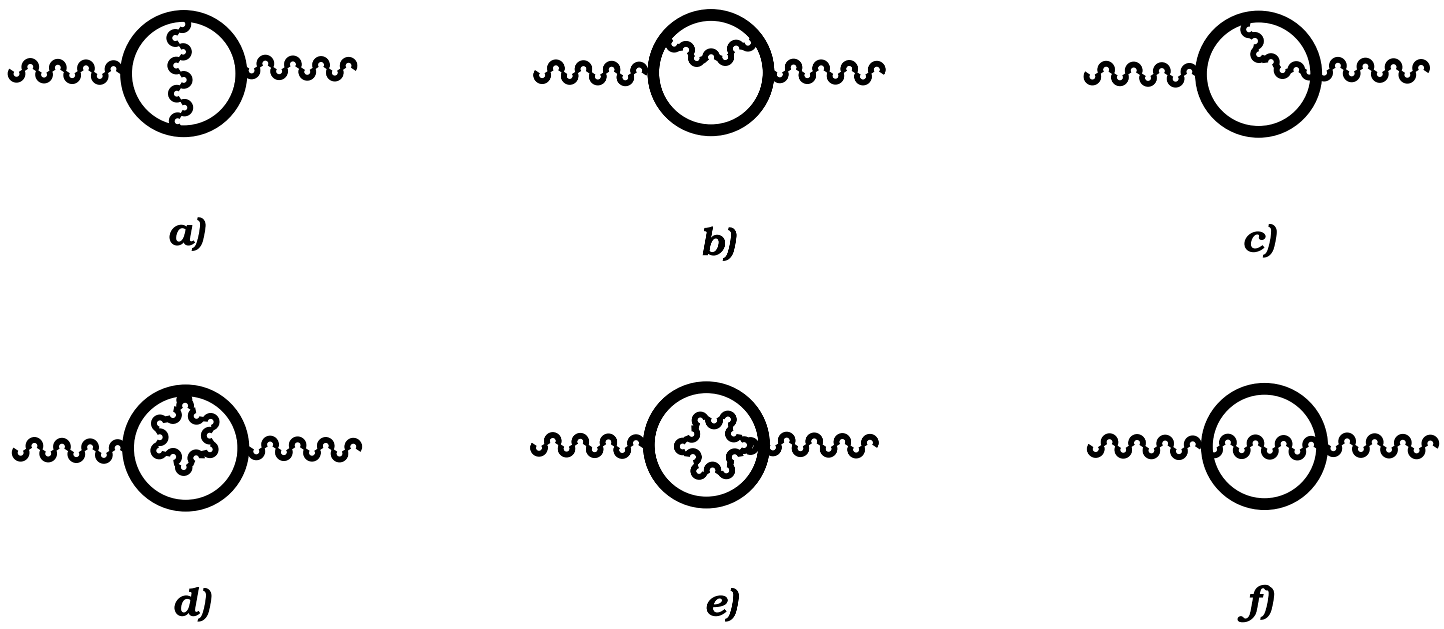
$$
\left\langle T_{zz}^{(\mathrm{reg})} \right\rangle_Y = -\frac{1}{12}[\partial^2\varphi - \frac{1}{2}(\partial\varphi)^2] + \mathcal{O}(\varphi^5) \qquad (1)
$$

tremendous cancellation of diagrams due to diffeomorphism invariance

$$
\frac{q}{b^2} = \frac{1}{b_0^2} - \frac{1}{6} + \mathcal{O}(b_0^2) \quad \Rightarrow \quad q = 1 + \mathcal{O}(b_0^4)
$$

9. Cancellation of logs at two loops

Computation of $S_{\text{eff}}^{(2)}$



wavy lines are now associated with φ.

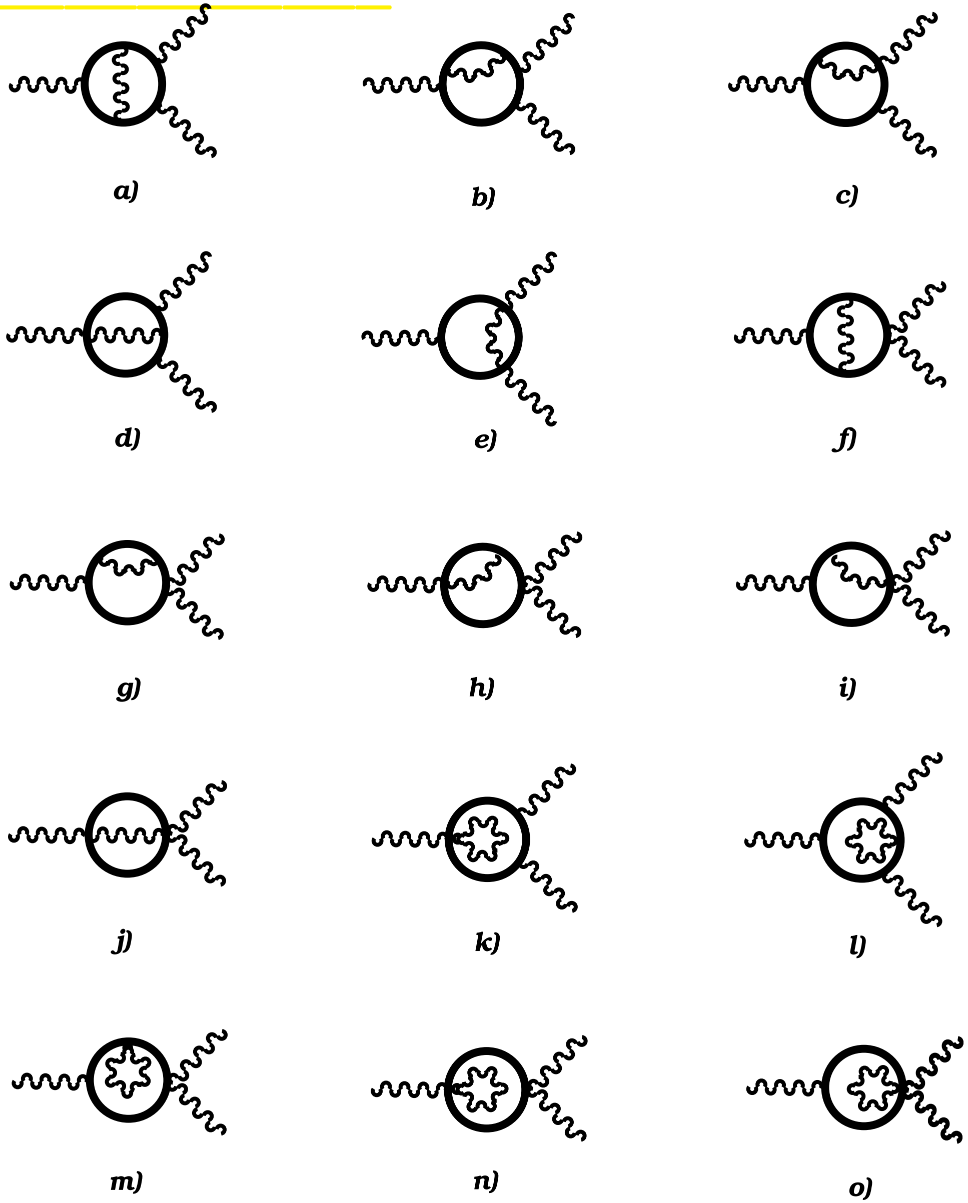
Combinatorics: $a) + 2b) - 4c) - d) + e) + f)$. The result

$$S_{\text{eff}}^{(2)} = \frac{1}{16\pi b^2}\int \partial_a\varphi\partial_a\varphi \quad \frac{1}{b^2} = \frac{1}{b_0^2} - \frac{1}{6} - 5b_0^2 + \mathcal{O}(b_0^2) \quad b_0^2 = \frac{6}{26-d}$$

Cancellation of IR divergences (theorem due to quadratic S_{eff}).

Most important for maintaining conformal invariance.

Computation of $S_{\mathrm{eff}}^{(3)}$



Combinatorics: $3a) + 3b) - 6c) - 3d) + 3e) - f) - 2g) + 2h) + 2i) - j)$
$+ \frac{3}{2}k) - \frac{3}{2}l) + m) - \frac{1}{2}n) - \frac{1}{2}o)$

Computation of $S_{\text{eff}}^{(3)}$ (cont.)

The result (IR divergences again mutually cancel)

$$S_{\text{eff}}^{(3)} = -\frac{127}{80\pi}\int \varphi\partial_a\varphi\partial_a\varphi + \mathcal{O}(h)$$

nonvanishing because of quadratic divergences $\frac{1}{\Lambda^2}\times\Lambda^2 =$ finite

In covariant notations

$$S_{\text{eff}}^{(3)} = -\frac{127}{160\pi}\int R\frac{1}{\Delta}R\frac{1}{\Delta}R + \mathcal{O}(h)$$

S_{eff} differs from (massless) Liouville and is not quadratic in φ
$\Longrightarrow$ φ cannot be a primary field (a field redefinition required!).

$$\rho(\varphi) = \mathrm{e}^{\Phi(\varphi)} \qquad\qquad \Phi \text{ is primary}$$

The measure for path-integration over Φ is generically nonlinear but Jacobian for $\varphi \to \Phi(\varphi)$ is inessential because of no derivatives

CFT consideration

Generically we get to all orders in b_0^2 (for $\hat{g}_{ab} = \delta_{ab}$)

$$S_{\text{eff}} = \frac{1}{16\pi b^2} \int f(\varphi) \partial_a \varphi \partial_a \varphi \qquad f(x) = 1 - \frac{127 b^2}{5} x$$

which is invariant under modified conformal transformation

$$\delta z = \xi(z) \qquad \delta \varphi = \frac{1}{\sqrt{f(\varphi)}} \partial \xi$$

nonlinear like for Polchinski-Strominger $\Longrightarrow$ φ is not primary

Introducing the new field which is now primary

$$\Phi(\varphi) = \int_0^{\varphi} \mathrm{d}x \sqrt{f(x)} \qquad \delta \Phi = \partial \xi$$

we write

$$S_{\text{eff}} = \frac{1}{16\pi \mathcal{B}^2} \int \partial_a \Phi \partial_a \Phi$$

where $\mathcal{B}^2$ accounts for the Jacobian from φ to Φ á la DDK
but it is *not* essential at one loop (Ambjørn, Y.M., Semenoff unpublished)

Function $f(x)$ may be nonuniversal (regularization dependent) but results seems to be the same (universality!)

Conclusion

Differences between the Nambu-Goto and Polyakov strings:

- Classical (perturbative) vacuum is stable only for $d < 2$.
  For $2 < d < 26$ the mean-field vacuum is stable instead

- Lilliputian strings for $d > 2$ versus Gulliver's strings for $d \leq 2$

- Gulliver's tools (inherited from QFT) are too coarse to deal with the Lilliputian scaling regime

- String susceptibility index differs: $1/2$ versus 1 for genus one

- Emergent action differs for nonsmooth φ

- 2D conformal invariance is maintained both classically and by fluctuations about the ground state (in both cases)

- The one-loop central charge of φ gets additional $6G$